5 General structures of linear ODEs (optiopnal)

Fact: A general solution to a n-th order ODE typically involve n indeterminate constants.

Example 5.1. A falling ball: $y'' = -g$ (gravitational constant). Initial conditions" initial position and velocity.

$y'' = \frac{d}{dt} y' = -g$

initial conditions

$\frac{dy}{dt} = y' = -gt + C_1$

$y(0) = y_0$

$y'(0) = v_0.$

$y(t) = -g\frac{t^2}{2} + C_1 t + C_2$

$\rightarrow C_2 = y_0 \quad C_1 = v_0$

Proposition 1 (structure of homogeneous linear ODEs)**.** If y_1, y_2 are two solutions of a homogeneous ODE, then for any constants C_1, C_2, $y = C_1\, y_1 + C_2\, y_2$ is also a solution.

n-th order homogeneous linear ODE

$f_n(x)\, y^{(n)} + f_{n-1}(x)\, y^{(n-1)} + \cdots f_1(x)\, y' + f_0(x)\, y = 0$

if y_1 is a solution $c\left(f_n y_1^{(n)} + f_{n-1} y_1^{(n-1)} + \cdots f_1 y_1' + f_0 y_1\right) = 0$

f_n

Example 5.2. Find all solutions of the ODE: $y'' - 3y' + 2y = 0$.

Proposition 2 (structure of linear ODEs)**.** A general solution y to a linear ODE has the form:

$$y = y_h + y_p,$$

where y_h is the general solution to the linear ODE's associated homogeneous linear ODE; y_p is a "particular solution" to the ODE itself.

Example 5.3. Find all solutions of the ODE: $y'' - 3y' + 2y = 2$.

We found from Eq. 5.2. that

$y_h = C_1 e^{t} + C_2 e^{2t}$

$y_p = 1$ is a particular solution

the general solution to this ODE is

$y = 1 + C_1 e^{t} + C_2 e^{2t}$

$\square$

If y_1 is a solution to

$$C\left(f_n y_1^{(n)} + f_{n-1} y_1^{(n-1)} + \cdots + f_1 y_1' + f_0 y_1\right) = 0$$

$$\Leftrightarrow \quad f_n ((C y_1)^{(n)}) + f_{n-1}(C y_1)^{(n-1)} + \cdots + f_1 (C y_1)' + f_0 (C y_1) = 0$$

$\Rightarrow$ $C y_1$ is also a solution to the homogeneous linear ODE

If y_1, y_2 are both solutions to the homogeneous linear ODE

$$+\begin{cases} f_n y_1^{(n)} + f_{n-1} y_1^{(n-1)} + \cdots f_1 y_1' + f_0 y_1 = 0 \\ f_n y_2^{(n)} + f_{n-1} y_2^{(n-1)} + \cdots + f_1 y_2' + f_0 y_2 = 0 \end{cases}$$

$$f_n (y_1^{(n)} + y_2^{(n)}) + \cdots\cdots \quad f_0 (y_1 + y_2)) = 0$$

$$\Leftrightarrow \quad f_n ((y_1 + y_2)^{(n)} + \cdots \quad f_0 (y_1 + y_2)) = 0$$

$\rightarrow$ $y_1 + y_2$ is also a solution to the homogeneous linear ODE.

When $n=1$, y_1, y_2 are not proportional

$$y_2 \neq C y_1$$

then $\underline{y = C_1 y_1 + C_2 y_2}$ is another soln to the homogeneous 2nd order linear ODE

C_1, C_2 are arbitrary

so this gives the general solution to the homogeneous linear ODE

(If $y_2 = c\,y_1$ then $y = C_1 y_1 + C_2 y_2 = \underbrace{(C_1 + c\,C_2)}\,y_1$

actually there is really only one arbitrary constant)

More generally, if $y_1, y_2, \cdots y_n$ are "linearly independent" solutions to a n-th order homologeneous linear ODE, then the general solution

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$$

E.g. $y'' - 3y' + 2y = 0$ —— $(*)$

try possible solutions:

reasonable guess: $y = e^{\lambda x}$

then $y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

plug in $(*)$

$$(\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x}) = 0$$

$$e^{\lambda x}(\lambda^2 - 3\lambda + 2) = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$$

so e^t, e^{2t} are both solutions to the ODE $(*)$

so the general solution to $(*)$ is

$$y = C_1 e^t + C_2 e^{2t}$$

$\square$

If y_1, y_2 are solutions to a linear n-th order ODE:

$$f_n y^{(n)} + f_{n-1} y^{(n-1)} + \cdots f_1 y' + f_0 y = g \qquad (*)$$

i.e.
$$\begin{cases} f_n y_1^{(n)} + f_{n-1} y_1^{(n-1)} + \cdots f_1 y_1' + f_0 y_1 = g \\ f_n y_2^{(n)} + f_{n-1} y_2^{(n-1)} + \cdots f_1 y_2' + f_0 y_2 = g \end{cases}$$

$$-$$

$$f_n (y_1^{(n)} - y_2^{(n)}) + \cdots + f_1 (y_1' - y_2') + f_0 (y_1 - y_2) = 0$$

$$f_n ((y_1 - y_2)^{(n)} + \cdots + f_1 (y_1 - y_2)' + f_0 (y_1 - y_2) = 0$$

so $y_h = y_1 - y_2$ is a solution to the associated homogeneous linear ODE to

i.e. $f_n y^{(n)} + \cdots f_0 y = 0$

A general homogeneous linear ODE with constant coefficients can be solved by a standard algorithm:

Trial function $y = e^{\lambda t}$

$\rightarrow \quad a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots a_n = 0$

$\Rightarrow$ solve for λ

$$a_n (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots Q_1^{m_1} \cdots \cdots = 0$$

irreducible quadratic function

$e^{\lambda_1 x}, x e^{\lambda t} \cdots x^{n-1} e^{\lambda t}$

combination of sin and cos function